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Tetrad, Connection, and Metric as Independent Variables in Lagrangians of Micromorphic Continua Models

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In continuous media theory, models with a tetrad, a metric, or a connection field as the independent variable are widely used. Unification of these formalisms is presented. Models with a Lagrangian dependent on tetrad, connection, and metric fields treated as independent variables are investigated. The tetrad and the connection play the role of dynamic variables, but the metric is a nondynamic one. This means there are no derivatives of the metric in the Lagrangian. In a Polyakov-like way, as in string theory, the metric is eliminated from the Lagrangian and field equations. The Lagrangian takes a simple square-root form as the Nambu Lagrangian. It connects in a sense Lagrangians from the $Gl(n, R)$-invariant and Kijowski theories. The distinguished solutions for a symmetric connection are semisimple Lie groups. The model gives the possibility of simultaneous description of fields of different natures and can be applied in the description of continuous media with complicated internal structure and in external fields.

1. INTRODUCTION

The mathematical description of anisotropic continuous media, for example, composites, is a difficult problem, because of the need to take into account the entire micromorphic structure of the media. On the other hand, solvable mathematical models which can take into account this structure and possible interaction with external fields are important in the descriptions of materials as well as in the calculation of their mechanical properties. There is a need to consider the internal micromorphic structure also from the geometrical point of view. In some geometrical models (Slawianowski, 1990;

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Trzęsowski and Slawianowski, 1990) of continuous media the independent variable is the tetrad field.

In the classical formulation of Einstein's theory of relativity the role of the gravitational potential is played by the metric. There exist, however, other formulations of this theory where the role of the gravitational potential is played by the dynamic connection (Kijowski, 1978) or the tetrad field (Slawianowski, 1985; Makaruk, 1993, 1994). It was proved that these two theories are equivalent to the Einstein theory. In the case of the tetrad theory the metric is built of the tetrad field and the numerical metric $\eta_{AB} = \mathrm{diag}(1, -1, -1, -1)$ using the recipe $g_{\mu\nu} = \eta_{AB}\, e^A{}_\mu e^B{}_\nu$. We can also find many other field theories in which the independent variables are a metric, a connection, or a tetrad field. Every formalism has its own advantages and disadvantages. In one of these theories, the $Gl(n, \mathbb{R})$-invariant theory based on the dynamic tetrad field, there exists a class of distinguished solutions in the form of semisimple Lie algebras.

In this paper we consider a model with Lagrangian dependent on a tetrad, a connection, and a metric treated as independent variables. The tetrad and the connection play the roles of dynamic variables, but the metric is a nondynamic one. The method of description of the problem and the way of finding solutions of the field equations are similar to those in the $Gl(n, \mathbb{R})$-invariant theory.

1.1. *Gl(n*, R)-Invariant Theory

Let us recall briefly the $Gl(n, \mathbb{R})$-invariant theory. The unique dynamic variable in the theory is a tetrad field. The motivation for such a theory comes from two sources. One is the existing tetrad description of gravitation. This description, however, employs an internal metric. Motivation also comes from the description of a continuum in terms of tetrad fields, but in which no length scale is distinguished. In other words, in this description there is no distinguished metric tensor. *A posteriori* we can note that the existence of semisimple Lie groups as canonical solutions was additional motivation for developing the theory.

An interesting problem appears: Is the existence of solutions of this kind an exceptional feature of this theory, or are there other theories of similar type possessing this feature? We will show that the answer is positive for the theory with a tetrad, a connection, and a metric treated as independent variables which is introduced in this paper.

The $Gl(n, \mathbb{R})$-invariant theory has two different formulations: a classical one and a Polyakov-like one. The basic object for the classical formulation of the theory is a manifold equipped only with its differential structure. For such a manifold there exist the following natural bundles: the tangent bundle and the bundles that descend from the tangent bundle—the frame bundle, which is a principal bundle with the group $Gl(n, \mathbb{R})$ acting transitively on fibers of the bundle, the dual bundle of coframes, and tensor bundles, which are built as the bundles associated to the frame bundle (among them are the tangent and the cotangent bundle). The basic field considered in the theory is the frame field, i.e., a section of the frame bundle. The action for this theory is built using only this field and its dual. This action, being a scalar density, has the characteristic form of a square root of a second-rank tensor built of the frame field in the following form:

$$\mathcal{L} := \sqrt{|\det \gamma_{ij}|} \tag{1}$$

$$\gamma_{ij} := \alpha S^k{}_{il} S^l{}_{jk} + \beta S^k{}_{ik} S^l{}_{jl} \tag{2}$$

$$S^i{}_{jk} := e^i{}_A e^A{}_{[j,k]} \tag{3}$$

where $e^i{}_A$ and $e^A{}_i$ are components of the frame and coframe field, respectively, $A, B, C, \ldots$ are anholonomic indices, $i, j, k, \ldots$ are holonomic indices, $S^i{}_{jk}$ is the torsion of the teleparallelism connection for a frame field, γ is a tensor playing the role of a metric, and $4S^k{}_{il}S^l{}_{jk}$ is an object called in this theory a Killing tensor because its form is analogous to the biinvariant Killing metric defined on group manifolds.

The Lagrangian depends on a tetrad (the frame field) and its derivatives by the torsion of the teleparallelism connection. Slawianowski introduced in this theory the generalized momentum H:

$$H_A{}^{ij} := \frac{\partial \mathcal{L}}{\partial e^A{}_{i,j}} = \frac{\partial \mathcal{L}}{\partial S^p{}_{ij}} e^p{}_A \tag{4}$$

$$H_k{}^{ij} := e^A{}_k H_A{}^{ij} \tag{5}$$

Field equations of the $Gl(n, \mathbb{R})$-invariant theory obtained by variation of the frame field are written in the covariant form using the teleparallelism connection (Slawianowski, 1991):

$$\nabla_j H_k{}^{ij} = -2H_k{}^{ij} S^l{}_{lj} \tag{6}$$

Slawianowski (1992) also introduced two other definitions for the formulation of the theorem concerning solutions of the field equations:

- A tetrad field is *Killing-nonsingular* iff the Killing tensor for this field is nondegenerate.
- A tetrad field is *closed* iff $\nabla S = 0$; in other words, the torsion tensor is invariant with respect to the parallel transport along the field e.

Slawianowski (1992) proved the following theorem:

Theorem. Every closed Killing-nonsingular tetrad field is a solution of field equations (6) of the $Gl(n, \mathrm{R})$-invariant theory.

This theorem shows that the semisimple Lie groups are the solutions of the theory. Components of the torsion tensor are in this case equal to the structure constants of a corresponding Lie algebra (neglecting factors of the type 1/2). For Lie groups the trace for structure constants $C^i{}_{ai}$ is equal to zero and this is equivalent to vanishing of the trace for the torsion tensor.

The Killing tensor is identified with the Killing metric of the Lie algebra. In the case of the Lie algebra Killing-nonsingularity is equivalent to the semisimplicity condition. Semisimple Lie algebras are canonical solutions of the theory.

The second version of the $Gl(n, \mathrm{R})$-invariant theory is a Polyakov-like formulation. The action in this theory plays a role similar to the role played by the Nambu action in the string theory. Nambu equations are classically equivalent to the equations implied by the Polyakov action. Similar equivalence for the two formulations of the $Gl(n, \mathrm{R})$-invariant theory was proved by Slawianowski (1992). Analogously to the method used by Polyakov, he introduced an action with the Lagrangian

$$\mathcal{L} = g^{ij}\gamma_{ij}\sqrt{g} + \Lambda\sqrt{g} \tag{7}$$

where Λ is the cosmological constant.

Variation of the metric leads to an algebraic dependence of g_{ij} on γ_{ij}, which for an appropriate choice of the constant Λ is of the form

$$g_{ij} = \gamma_{ij} \tag{8}$$

Substitution of (8) into (7) changes the Lagrangian density to the form (1) (modulo a constant). Obviously, this theory, being equivalent to the previous formulation, has canonical solutions in the form of local semisimple Lie groups.

2. DESCRIPTION OF SPACE-TIME WITH METRIC, CONNECTION, AND TETRAD TREATED AS INDEPENDENT FIELDS

Let us consider a theory with a metric, a connection, and a tetrad treated as independent variables. The Lagrangian is built of the following objects:

- metric g_{AB}
- tetrad $e^A{}_i$ (here cotetrad)
- dynamic connection $\Gamma^i{}_{jk}$

which are subject to variation.

The full connection of the system is 1/2 of the sum of the connection $\Gamma^i{}_{jk}$ and the teleparallelism connection for the tetrad field. The metric g_{AB} is not dynamic, which means the Lagrangian does not contain its derivatives. This approach is analogous to the Polyakov-like theory. From

$$\frac{\partial L}{\partial g_{AB}} = 0 \tag{9}$$

we obtain the dependence of g_{AB} on e_i^A and $\Gamma^i{}_{jk}$.

The Lagrangian density in this theory is of the form

$$\mathcal{L} = a\mathcal{L}_1 + b\mathcal{L}_2 + \Lambda\sqrt{g} \tag{10}$$

where

$$\begin{aligned} \mathcal{L}_1 &= g^{ij}\gamma_{ij}(e, \Gamma)\sqrt{g} \\ \mathcal{L}_2 &= g^{ij}R_{ij}(\Gamma)\sqrt{g} \end{aligned} \tag{11}$$

where

$$\begin{aligned} \gamma_{ij} &= S^k{}_{il}S^l{}_{jk} + \gamma S^k{}_{ik}S^l{}_{jl} \\ S^i{}_{jk} &= \tfrac{1}{2}[e^i{}_A e^A{}_{[j,k]} + S^i{}_{jk}] = \tfrac{1}{2}[e^i{}_A e^A{}_{[j,k]} + \Gamma^i{}_{[jk]}] \\ g_{ij} &= g_{AB}e^A{}_i e^B{}_j \\ g^{ij}g_{kj} &= \delta^i_k \end{aligned}$$

$R_{ij}(\Gamma)$ is the Ricci tensor for the connection Γ (the teleparallelism connection gives the known contribution to R_{ij}), and γ_{ij} is built of the two Weitzenböck invariants. The third Weitzenböck invariant is of different form and could not be included in the theory in the same manner as these two.

The first step in our considerations is the variation of the action with respect to g_{AB}:

$$\delta\mathcal{L}_1 = \delta g^{ij}(\delta g_{AB})\gamma_{ij}\sqrt{g} + g^{kl}\gamma_{kl}\frac{\partial(\sqrt{g})}{\partial g_{ij}}\frac{\partial g_{ij}}{\partial g_{AB}}\delta g_{AB} \tag{12}$$

Then we express the variation of g_{ij} in terms of the variation of g_{AB}. We write this in detail here, because a similar scheme will be used in our further investigations:

$$\begin{aligned} \delta \mid g^{ij}g_{klj} &= \delta^i_k \\ \delta g^{ij}g_{kj} + g^{ij}\delta g_{kj} &= 0 \mid g^{kl} \\ \delta g^{ij}\delta^l_j + g^{ij}g^{kl}\delta g_{jk} &= 0 \\ \delta g^{ij} &= -g^{ik}g^{jl}\delta g_{kl} = -g^{ik}g^{jl}\delta(g_{AB}e^A_k e^B_l) = -g^{ik}g^{jl}\delta g_{AB}(e^A_k e^B_l) \end{aligned}$$

$$\frac{\partial(\sqrt{g})}{\partial g_{ij}} = \frac{1}{2\sqrt{g}} g g^{ij} = \frac{\sqrt{g}}{2} g^{ij}$$

$$\frac{\partial g_{ij}}{\partial g_{AB}} = \frac{\partial(g_{CD} e^C{}_i e^D{}_j)}{\partial g_{AB}} = e^A{}_i e^B{}_j$$

Analogously, using the derived relations, we can calculate the desired variations of all components of the Lagrangian density:

$$\delta\mathcal{L}_1 = \delta g_{AB}\sqrt{g}[\tfrac{1}{2} g^{kl}\gamma_{kl} g^{AB} - g^{ik} g^{jl} e^A{}_k e^B{}_l \gamma_{ij}] \tag{13}$$

$$\delta\mathcal{L}_2 = \delta g_{AB}\sqrt{g}[\tfrac{1}{2} g^{kl} R_{kl} g^{AB} - g^{ik} g^{jl} e^A{}_k e^B{}_l R_{ij}] \tag{14}$$

The variation of the complete Lagrangian density is of the form

$$\delta\mathcal{L} = (\delta g_{AB})\sqrt{g}\{(a\gamma_{kl} + bR_{kl})(\tfrac{1}{2} g^{kl} g^{AB} - g^{ik} g^{jl} e^A{}_i e^B{}_j) + \tfrac{1}{2}\Lambda g^{AB}\} \tag{15}$$

The equations following from $\delta\mathcal{L}/\delta g_{AB} = 0$ are of the form

$$(a\gamma_{kl} + bR_{kl})(\tfrac{1}{2} g^{kl} g^{AB} - g^{ik} g^{jl} e^A{}_i e^B{}_j) + \tfrac{1}{2}\Lambda g^{AB} = 0 \tag{16}$$

It is easier to consider the necessary condition for (16), which we obtain by taking the trace, i.e., by contracting with g_{AB}:

$$\left(\frac{n}{2} - 1\right) g^{kl}(a\gamma_{kl} + bR_{kl}) + \Lambda\frac{n}{2} = 0 \tag{17}$$

This condition can be written in the form

$$g^{kl}(a\gamma_{kl} + bR_{kl}) = -\frac{n}{n-2}\Lambda \tag{18}$$

Therefore

$$(a\gamma_{kl} + bR_{kl}) = \alpha g_{kl} \tag{19}$$

Substitution of (19) into (18) gives a condition involving α:

$$n\alpha = -\frac{n}{n-2}\Lambda \tag{20}$$

Normalizing α, $\alpha = 1$, we obtain for the cosmological constant $\Lambda = 2 - n$. Then

$$g_{kl} = a\gamma_{kl} + bR_{kl} \tag{21}$$

Here the metric plays a role analogous to the role of the metric in the string theory in Polyakov's formulation. Substitution of this metric into the Lagrangian gives

$$\mathcal{L} = g^{kl}(a\gamma_{kl} + bR_{kl})\sqrt{g} + \Lambda\sqrt{g}$$
$$= 2\sqrt{\det[a\gamma_{kl}(e, \Gamma) + bK_{kl}(\Gamma)]} \tag{22}$$

where R_{ij} is the Ricci tensor, K_{ij} is the symmetric part of the Ricci tensor, and a, b are constants. This is a theory with a dynamic tetrad and a dynamic connection. It has a Lagrangian with the characteristic form of a square root of a second-rank tensor. In general this tensor is not symmetric, however. This way, proceeding in the spirit of Polyakov's scheme, we eliminated from consideration the metric, which was from the very beginning a nondynamic variable in the theory. The general form of the Lagrangian density is very interesting in itself, in the sense that it connects the Lagrangians from the Slawianowski (1985) and Kijowski (1978) theories.

2.1. Model with Symmetric Connection

Let us make some limiting assumptions, which are of a similar kind to those in Kijowski's theory. Let us assume that the dynamic connection $\Gamma^i{}_{jk}$ is symmetric. For a symmetric connection the metric γ_{ij} built from the torsion tensor depends only on the tetrad field and does not depend on the connection. The Lagrangian density can be written in the form

$$\mathcal{L} = 2\sqrt{|a\gamma_{ij}(e) + bK_{ij}(\Gamma)|} \tag{23}$$

The field equations for the tetrad are given by

$$\frac{\partial \mathcal{L}}{\partial e^A{}_i} = \partial_j\left(\frac{\partial \mathcal{L}}{\partial e^A{}_{i,j}}\right) \tag{24}$$

Let us introduce the generalized momentum H defined in the following way:

$$H_A{}^{ij} := \frac{\partial \mathcal{L}}{\partial e^A{}_{i,j}} \tag{25}$$

$$H_A{}^{ij} \overset{tel}{=} (\partial \mathcal{L}/\partial S^p{}_{ji}) e^p_A \tag{26}$$

$$H_A{}^{ij} \overset{tel}{=} H_A{}^{ij}(S(e), e, K(\Gamma)) \tag{27}$$

The momentum $H_A{}^{ij}$ defined above is antisymmetric in the holonomic indices:

$$H_A{}^{(ij)} = 0 \tag{28}$$

The field equations obtained by variation of the Lagrangian (23) are of the form

$$\partial\mathcal{L}/\partial e^A{}_i = -(\partial\mathcal{L}/\partial \overset{tel}{S}{}^p{}_{qr})e^p_A \overset{tel}{S}{}^i{}_{qr} \tag{29}$$

$$H_A{}^{ij}{}_{,j} = -(\partial\mathcal{L}/\partial \overset{tel}{S}{}^p{}_{qr})e^p_A \overset{tel}{S}{}^i{}_{qr} \tag{30}$$

The next step is the calculation of a covariant derivative of the momenta H with respect to any connection $\tilde{\Gamma}$:

$$H_A{}^{ij}{}_{\tilde{;}j} = (\overset{\tilde{\Gamma}}{S}{}^i{}_{jl} - \overset{tel}{S}{}^i{}_{jl})H_A{}^{lj} + 2\overset{\tilde{\Gamma}}{S}{}^j{}_{jl}H_A{}^{il} \tag{31}$$

Because the quantities $H_A{}^{ij}$ were defined earlier in (25) as generalized momenta, the equalities (31) are field equations for the Lagrangian density (23).

If $\tilde{\Gamma} = \overset{tel}{\Gamma}$, the field equations (31) simplify to the form

$$H_A{}^{ij}{}_{;j} = 2\overset{tel}{S}{}^j{}_{jl}H_A{}^{il} \tag{32}$$

For a Lie group the following relations are satisfied:

$$S^j{}_{jl} = 0 \tag{33}$$

$$\overset{tel}{\nabla} S = 0 \tag{34}$$

If additionally the condition

$$\overset{tel}{\nabla} K_{ij}(\Gamma) = 0 \tag{35}$$

is satisfied; then this part of the field equations is satisfied. The rest of the field equations remain to be satisfied. Let us consider the equations

$$\frac{\partial\mathcal{L}}{\partial\Gamma^i{}_{jk}} = \partial_l\left(\frac{\partial\mathcal{L}}{\partial\Gamma^i{}_{jk,l}}\right) \tag{36}$$

$$\frac{\partial\mathcal{L}}{\partial\Gamma^i{}_{jk}} = \frac{\partial\mathcal{L}}{\partial K_{rs}}\frac{\partial K_{rs}}{\partial\Gamma^i{}_{jk}}$$

The next step is the calculation of the quantity $\partial K_{rs}/\partial\Gamma^i{}_{jk}$. We introduce the generalized momentum defined as follows:

$$P_i^{jkl}(S(e), K(\Gamma)) := \frac{\partial\mathcal{L}}{\partial\Gamma^i{}_{jk,l}} \tag{37}$$

$$P_i^{jkl} = \frac{\partial\mathcal{L}}{\partial K_{rs}}\frac{\partial K_{rs}}{\partial\Gamma^i{++}_{jk,l}} \tag{38}$$

Then we calculate $\partial K_{rs}/\partial \Gamma^i{}_{jk,l}$ and $\partial K_{rs}/\partial \Gamma^i{}_{jk}$. After some manipulations we obtain

$$\frac{\partial \mathcal{L}}{\partial \Gamma^i{}_{jk}} = \delta_i^j \Gamma^k{}_{(rs)} \frac{\partial \mathcal{L}}{\partial K_{rs}} + \Gamma^l{}_{li} \frac{\partial \mathcal{L}}{\partial K_{jk}} - \Gamma^k{}_{ir} \frac{\partial \mathcal{L}}{\partial K_{rj}} - \Gamma^j{}_{ri} \frac{\partial \mathcal{L}}{\partial K_{rk}} \tag{39}$$

$$P_i^{jkl} = \delta_i^l \frac{\partial \mathcal{L}}{\partial K_{jk}} - \delta_i^j \frac{\partial \mathcal{L}}{\partial K_{kl}} \tag{40}$$

The field equations become

$$P_i^{jkl}{}_{,l} = \delta_i^j \Gamma^k{}_{(rs)} \frac{\partial \mathcal{L}}{\partial K_{rs}} + \Gamma^l{}_{li} \frac{\partial \mathcal{L}}{\partial K_{jk}} - \Gamma^k{}_{ir} \frac{\partial \mathcal{L}}{\partial K_{rj}} - \Gamma^j{}_{ri} \frac{\partial \mathcal{L}}{\partial K_{rk}} \tag{41}$$

Further we compute the covariant derivative of a generalized momentum with respect to any connection $\hat{\Gamma}^j{}_{kl}$. In general, this connection is different from all connections introduced earlier. In terms of the general connection, the field equations can be written in the form

$$\begin{aligned} P_i^{jkl}{}_{;\hat{l}} = {} & \delta_i^j [\Gamma^k{}_{(rs)} - \hat{\Gamma}^k{}_{(rs)}] \frac{\partial \mathcal{L}}{\partial K_{rs}} + [\Gamma^m{}_{mi} - \hat{\Gamma}^m{}_{mk}] \frac{\partial \mathcal{L}}{\partial K_{jk}} \\ & + [\Gamma^k{}_{mi} - \hat{\Gamma}^k{}_{mi}] \frac{\partial \mathcal{L}}{\partial K_{jm}} + 2\hat{S}^j{}_{im} \frac{\partial \mathcal{L}}{\partial K_{km}} + 2\hat{S}^l{}_{li} \frac{\partial \mathcal{L}}{\partial K_{jk}} \\ & - 2\delta_i^j \hat{S}^l{}_{lm} \frac{\partial \mathcal{L}}{\partial K_{km}} \end{aligned} \tag{42}$$

$$P_i^{jkl} = P_i^{jkl}(S(e), K(\Gamma)) \tag{43}$$

Next we take $\hat{\Gamma} = \Gamma$. As a result the field equations simplify to

$$P_i^{jkl}{}_{;l} = 2S^j{}_{im} \frac{\partial \mathcal{L}}{\partial K_{km}} + 2S^l{}_{li} \frac{\partial \mathcal{L}}{\partial K_{jk}} - 2\delta_i^j S^l{}_{lm} \frac{\partial \mathcal{L}}{\partial K_{km}} \tag{44}$$

where the semicolon denotes the covariant derivative with respect to the connection Γ.

For the symmetric connection Γ the field equations take the form

$$P_i^{jkl}{}_{;l} = 0 \tag{45}$$

These field equations are satisfied, e.g., in the case when the following relations hold:

$$\overset{\Gamma}{\nabla} S = 0 \tag{46}$$

$$\overset{\Gamma}{\nabla} K = 0 \tag{47}$$

Let us consider the conditions of the given form in the Lagrangian (23). We compute the quantities $\partial \mathcal{L} / \partial \overset{tel}{S^p}_{ji}$, which are present in the definition of the generalized momenta $H_A{}^{ij}$ in (25):

$$\partial \mathcal{L} / \partial \overset{tel}{S^p}_{ji} = a\sqrt{\det(a\gamma + bK)}(a\gamma + bK)^{jn} S^i{}_{np} \tag{48}$$

$$P_i^{jkl} = \frac{b}{2}\sqrt{\det(a\gamma + bK)}(a\gamma + bK)^{jk}\delta^l{}_i - \frac{b}{2}\sqrt{\det(a\gamma + bK)}(a\gamma + bK)^{kl}\delta^j{}_i \tag{49}$$

By analogy with Kijowski's theory, we define the metric as a canonical momentum conjugate to the connection, by the formulas

$$\frac{\partial \mathcal{L}}{\partial K_{ji}} =: \sqrt{g}g^{ij} = \frac{b}{2}\sqrt{\det(a\gamma + bK)}(a\gamma + bK)^{ij} \tag{50}$$

$$\partial \mathcal{L} / \partial \overset{tel}{S^p}_{ji} = \frac{a}{b}\sqrt{g}g^{jn} S^i{}_{np} \tag{51}$$

$$H_A{}^{ij} = \frac{a}{b}\sqrt{g}g^{jm} S^i{}_{mp} e^p_A \tag{52}$$

$$P_i^{jkl} = \delta_i^l \sqrt{g}g^{jk} - \delta_i^j \sqrt{g}g^{kl} \tag{53}$$

The equation

$$H_A{}^{ij}{}_{\underset{;j}{tel}} = 0 \tag{54}$$

is satisfied when the teleparallelism connection is metric w.r.t. the metric g_{ij} introduced this way and $\overset{\Gamma}{\nabla} S = 0$. These conditions are satisfied by Lie groups. The fact that Γ is metric is a conclusion in our model, not an assumption. This follows from equation (35). The condition of nondegeneracy of the Lagrangian forces these groups to be semisimple Lie groups.

3. SUMMARY

A theory with a Lagrangian dependent on three variables, a tetrad, a connection, and a metric, was formulated and investigated. The tetrad field and the connection were treated as independent variables; the tetrad field and the connection are dynamic variables, the metric is a nondynamic variable. The theory is of Polyakov type. Proceeding in the way typical for Polyakov-like theories, one can express the metric by residual variables after varying the Lagrangian with respect to the metric (as in string theory). As a result the metric disappears from further considerations. The resulting Lagrangian depends only on the tetrad and the connection. It can be expressed in the form of a square root of the determinant of a second-rank covariant tensor. Then this model was considered with the connection assumed to be symmetric. The Lagrangian density in this case takes the form (23). It was proved that in this case semisimple Lie groups are distinguished solutions, as in the $Gl(n, \mathbb{R})$-invariant theory.

The model is a generalization of models in which the independent variable is one of the fields: Slawianowski's theory, in which the only independent variable is the tetrad field, and the Kijowski's theory, in which the only independent variable is the dynamic connection. The formal relationship of all these models was presented.

The model is highly general and gives the possibility of finding exact solutions of the equations of motion in the description of anisotropic continuous media. There are degrees of freedom of three different types in the presented formalism. It can be applied to anisotropic continuous media in external fields.

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